Screening scenario for cosmological constant in de Sitter solutions, phantom-divide crossing and finite-time future singularities in non-local gravity

Reference: K. Bamba, S. Nojiri, S. D. Odintsov and M. Sasaki, arXiv:1104.2692 [hep-th].

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I. Introduction

- Recent observations of Supernova (SN) Ia confirmed that the current expansion of the universe is accelerating.

  [Perlmutter *et al*. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999)]

  [Riess *et al*. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998)]

  [Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

- There are two approaches to explain the current cosmic acceleration. **[Copeland, Sami and Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006)]**

  [Tsujikawa, arXiv:1004.1493 [astro-ph.CO]]

< Gravitational field equation >

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Gravity **Matter**

$G_{\mu\nu}$: Einstein tensor

$T_{\mu\nu}$: Energy-momentum tensor

$\kappa^2 \equiv 8\pi / {M_{\rm Pl}}^2$

$M_{\rm Pl}$: Planck mass

(1) **General relativistic approach** $\longrightarrow$ **Dark Energy**

(2) **Extension of gravitational theory**

(1) General relativistic approach

- Cosmological constant
- Scalar fields: X matter, Quintessence, Phantom, K-essence, Tachyon.
- Fluid: Chaplygin gas

$F(R)$: Arbitrary function of the Ricci scalar R

(2) Extension of gravitational theory

[Capozziello, Cardone, Carloni and Troisi, Int. J. Mod. Phys. D 12, 1969 (2003)]

- $F(R)$ gravity [Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]
- Scalar-tensor theories [Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]
- Ghost condensates [Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405, 074 (2004)]

$\mathcal{G}$: Gauss-Bonnet term

- Higher-order curvature term
  - $f(\mathcal{G})$ gravity

T : torsion scalar

- DGP braneworld scenario [Dvali, Gabadadze and Porrati, Phys. Lett B 485, 208 (2000)]
- $f(T)$ gravity [Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)] [Linder, Phys. Rev. D 81, 127301 (2010) [Erratum-ibid. D 82, 109902 (2010)]]
- Galileon gravity [Nicolis, Rattazzi and Trincherini, Phys. Rev. D 79, 064036 (2009)]

Non-local gravity ⇐ produced by quantum effects

[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]

- There was a proposal on the solution of the cosmological constant problem by non-local modification of gravity.

[Arkani-Hamed, Dimopoulos, Dvali and Gabadadze, arXiv:hep-th/0209227]

→ Recently, an explicit mechanism to screen a cosmological constant in non-local gravity has been discussed.

[Nojiri, Odintsov, Sasaki and Zhang, Phys. Lett. B 696, 278 (2011)]

Recent related reference: [Zhang and Sasaki, arXiv:1108.2112 [gr-qc]]

- It is known that so-called matter instability occurs in $F(R)$ gravity.

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

→ This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.

⇒ It is important to examine whether there exists the curvature singularity, i.e., "**the finite-time future singularities**" **in non-local gravity**.

- We investigate **de Sitter solutions in non-local gravity**.

  ⇨ We examine **a condition to avoid a ghost** and discuss **a screening scenario for a cosmological constant in de Sitter solutions**.

- We explicitly demonstrate that **three types of the finite-time future singularities can occur in non-local gravity and explore their properties**.

  → It is shown that **the addition of an R^2 term can cure the finite-time future singularities in non-local gravity**.

II. de Sitter solution in non-local gravity

A. Non-local gravity

$g = \det(g_{\mu\nu})$ $g_{\mu\nu}$: Metric tensor

< Action >

f : Some function Λ : Cosmological constant

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1 + \underline{f(\Box^{-1}R)}\right) - 2\Lambda\right] + \mathcal{L}_{\mathrm{matter}}\left(Q; g\right)\right\}$$

$\boxed{\text{Non-local gravity}}$

By introducing two scalar fields η and ξ , we find

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1 + f(\eta)\right) + \xi\,\underline{\left(\Box\eta - R\right)} - 2\Lambda\right] + \mathcal{L}_{\mathrm{matter}}\right\}$$

$$= \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1 + f(\eta)\right) - \partial_\mu\xi\partial^\mu\eta - \xi R - 2\Lambda\right] + \mathcal{L}_{\mathrm{matter}}\right\}$$

- By the variation of the action in the first expression over ξ , we obtain

$\underline{\Box\eta = R}$ (or $\eta = \Box^{-1}R$)

→ Substituting this equation into the action in the first expression, one re-obtains the starting action.

∇_μ : Covariant derivative operator

$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$: Covariant d'Alembertian

$\mathcal{L}_{\mathrm{matter}}\left(Q; g\right)$: Matter Lagrangian

Q : Matter fields

< Gravitational field equation >

$$0 = \frac{1}{2} g_{\mu\nu} \left[R \left(1 + f(\eta) - \xi \right) - \partial_\rho \xi \partial^\rho \eta - 2\Lambda \right] - R_{\mu\nu} \left(1 + f(\eta) - \xi \right)$$
$$+ \frac{1}{2} \left(\partial_\mu \xi \partial_\nu \eta + \partial_\mu \eta \partial_\nu \xi \right) - \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \left(f(\eta) - \xi \right) + \kappa^2 T_{\mathrm{matter}\,\mu\nu}$$

$$T_{\mathrm{matter}\,\mu\nu} \equiv - \left(2/\sqrt{-g} \right) \left(\delta \sqrt{-g} \mathcal{L}_{\mathrm{matter}} / \delta g^{\mu\nu} \right)$$

: Energy-momentum tensor of matter

- The variation of the action with respect to η gives

$$0 = \Box \xi + f'(\eta) R$$

$'$ (prime) : Derivative with respect to η

< Flat Friedmann-Lemaȋtre-Robertson-Walker (FLRW) metric >

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2$$

$a(t)$: Scale factor

- We consider the case in which the scalar fields η and ξ only depend on time.

→ Gravitational field equations in the FLRW background:

$$0 = -3H^2 \left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + \kappa^2 \rho_{\mathrm{m}}$$

$$0 = \left(2\dot{H} + 3H^2\right)\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(f(\eta) - \xi\right) - \Lambda + \kappa^2 P_{\mathrm{m}}$$

$\dot{} = \partial/\partial t$ $\quad H = \dot{a}/a$: Hubble parameter

ρ_{m} and P_{m} : Energy density and pressure of matter, respectively.

→ For a perfect fluid of matter: $T_{\mathrm{matter}\,00} = \rho_{\mathrm{m}}$

$T_{\mathrm{matter}\,ij} = P_{\mathrm{m}}\delta_{ij}$

< Equations of motion for η and ξ >

$$0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\eta)$$

$$R = 6\dot{H} + 12H^2$$

B. de Sitter solution

- We assume a de Sitter solution: $\boxed{H = H_0}$ H_0 : Constant

$\Rightarrow$ $\eta = -4H_0 t - \eta_0 \mathrm{e}^{-3H_0 t} + \eta_1$ $\eta_0,\ \eta_1$: Constants of integration

- We also suppose $\boxed{f(\eta) = f_0 \mathrm{e}^{\frac{\eta}{\beta}} = f_0 \mathrm{e}^{-\frac{4H_0 t}{\beta}}}$. $\eta_0 = \eta_1 = 0$

$\Rightarrow$ $\xi = -\frac{3f_0\beta}{3\beta - 4}\mathrm{e}^{-\frac{4H_0 t}{\beta}} + \frac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1$ $\xi_0,\ \xi_1$: Constants

- For the de Sitter space, a behaves as $a = a_0 \mathrm{e}^{H_0 t}$. a_0 : Constant

- For the matter with the constant equation of state $w_{\mathrm{m}} \equiv P_{\mathrm{m}}/\rho_{\mathrm{m}}$,

$\rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\mathrm{e}^{-3(w_{\mathrm{m}}+1)H_0 t}$ $\rho_{\mathrm{m}0}$: Constant

$\longrightarrow$ Putting $\xi_0 = 0$, we obtain

$$0 = -3H_0^2\left(1+\xi_1\right) + 6H_0^2 f_0 \left(\frac{2}{\beta} - 1\right)\mathrm{e}^{-\frac{4H_0 t}{\beta}} + \Lambda + \kappa^2 \rho_{\mathrm{m}0}\mathrm{e}^{-3(w_{\mathrm{m}}+1)H_0 t}$$

- For $\rho_{m0} = 0$, $\beta = 2$, $\xi_1 = -1 + \dfrac{\Lambda}{3H_0^2}$

For $\rho_m \neq 0$, $\beta = \dfrac{4}{3(1+w_m)}$, $f_0 = -\dfrac{\kappa^2 \rho_{m0}}{3H_0^2(1+3w_m)}$, $\xi_1 = -1 + \dfrac{\Lambda}{3H_0^2}$

$\longrightarrow$ There is a de Sitter solution.

$$\boxed{H_0^2 = \frac{\Lambda}{3(1+\xi_1)}}$$

⇨ **This means that the cosmological constant Λ is effectively screened by ξ .**

For $\Lambda = 0$, if we choose $\xi_1 = -1$, H_0 can be arbitrary.

$\longrightarrow$ Thus, H_0 can be determined by an initial condition.

⇨ Since H_0 can be small or large, the theory with the function $f(\eta) = f_0 e^{\frac{\eta}{\beta}} = f_0 e^{-\frac{4H_0 t}{\beta}}$ with $\beta = 2$ could describe **the early-time inflation or current cosmic acceleration**.

C. Condition to be free of ghost

- To examine the ghost-free condition, we make a conformal transformation to the Einstein frame:

$$\boxed{g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})} \qquad \Omega^2 = \frac{1}{1+f(\eta)-\xi}}$$

∗ A superscription (E) represents quantities in the Einstein frame.

$$R = \frac{1}{\Omega^2}\left[R^{(\mathrm{E})} - 6\left(\Box \ln\Omega + g^{(\mathrm{E})\,\mu\nu}\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega\right)\right]$$

$$\phi = \ln\Omega = -\frac{1}{2}\ln\left(1+f(\eta)-\xi\right)$$

$$S = \int d^4x\sqrt{-g^{(\mathrm{E})}}\left[\frac{1}{2\kappa^2}\left(R^{(\mathrm{E})} - 6g^{(\mathrm{E})\,\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \mathrm{e}^{2\phi}g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\eta - 2\mathrm{e}^{4\phi}\Lambda\right) + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q;\mathrm{e}^{2\phi}g^{(\mathrm{E})}\right)\right]$$

∗ We have discarded the $\Box\ln\Omega$ term because it is a total divergence.

- Instead of η and ξ, we may regard ϕ and η to be independent fields.

$$\longleftarrow \quad \xi = -\mathrm{e}^{-2\phi} + (1+f(\eta))$$

$$\boxed{S = \int d^4x\sqrt{-g^{(\mathrm{E})}}\left\{\frac{1}{2\kappa^2}\left[R^{(\mathrm{E})} - 6\nabla^\mu\phi\nabla_\mu\phi - 2\nabla^\mu\phi\nabla_\mu\eta - \mathrm{e}^{2\phi}f'(\eta)\nabla^\mu\eta\nabla_\mu\eta - 2\mathrm{e}^{4\phi}\Lambda\right] + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q;\mathrm{e}^{2\phi}g^{(\mathrm{E})}\right)\right\}}$$

- In order to avoid a ghost, the determinant of the kinetic term must be positive, which means

[Nojiri, Odintsov, Sasaki and Zhang, Phys. Lett. B 696, 278 (2011)]

$$\det \begin{vmatrix} 6 & 1 \\ 1 & \mathrm{e}^{2\phi} f'(\eta) \end{vmatrix} = 6\mathrm{e}^{2\phi} f'(\eta) - 1 > 0$$

$\longrightarrow$ This condition is assumed to be satisfied. In particular, $f'(\eta) > 0$ is a necessary condition.

$$\boxed{f'(\eta) > \frac{1}{6\mathrm{e}^{2\phi}} = \frac{1 + f(\eta) - \xi}{6} > 0}$$

: **The ghost-free condition**

- For $f(\eta) = f_0 \mathrm{e}^{\frac{\eta}{\beta}} = f_0 \mathrm{e}^{-\frac{4H_0 t}{\beta}}$

$$\xi = -\frac{3 f_0 \beta}{3\beta - 4} \mathrm{e}^{-\frac{4H_0 t}{\beta}} + \frac{\xi_0}{3H_0} \mathrm{e}^{-3H_0 t} - \xi_1 , \quad \xi_0 = 0 , \quad \beta = 2 , \quad \xi_1 = -1 + \frac{\Lambda}{3H_0^2}$$

$$\Longrightarrow \quad \frac{3}{4 + \frac{\Lambda}{3H_0^2 f_0} \mathrm{e}^{2H_0 t}} > 1$$

: **The ghost-free condition**

de Sitter universe is stable in a period

$$\frac{1}{2H_0}\ln\left(-\frac{3H_0^2 f_0}{\Lambda}\right) < t < \frac{1}{2H_0}\left[\ln 4 + \ln\left(-\frac{3H_0^2 f_0}{\Lambda}\right)\right]$$

The length of the ghost-free period

$$\Delta t = \frac{\ln 4}{2H_0} = \frac{\ln 2}{H_0} \simeq \frac{0.69}{H_0}$$

: The period is less than one *e*-folding time.

$\longrightarrow$ **This cannot give inflation in the early universe provided that the appearance of a ghost has to be avoided.**

III. Finite-time future singularities in non-local gravity

A. Finite-time future singularities

→ In the flat FLRW space-time, we analyze an asymptotic solution of the gravitational field equations in the limit of the time t_{s} when the finite-time future singularities appear.

- We consider the case in which the Hubble parameter is expressed as

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

h_{s} : Positive constant

q : Non-zero constant larger than -1 $(q > -1, q \neq 0)$

We only consider the period $0 < t < t_{\mathrm{s}}$.

- When $t \to t_{\mathrm{s}}$, $R = 6\dot{H} + 12H^2 \to \infty$

Scale factor

$$\boxed{a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}} - t)^{-(q-1)}\right]}$$

a_{s} : Constant

- By using $\ddot{\eta} + 3H\dot{\eta} = a^{-3} d\left(a^3 \dot{\eta}\right)/dt$ and $0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$,

$$\boxed{\eta = -\int^t \frac{1}{a^3}\left(\int^{\bar{t}} R a^3 d\bar{t}\right) dt}$$

η_{c} : Integration constant

- We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$.

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$: Non-zero constants

- By using $\ddot{\xi} + 3H\dot{\xi} = a^{-3} d\left(a^3 \dot{\xi}\right)/dt$ and $0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\eta)$,

$$\boxed{\xi = \int^t \frac{1}{a^3}\left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} R a^3 d\bar{t}\right) dt}$$

ξ_{c} : Integration constant

⇒ There are three cases.

(i) $[q > 1,\ \sigma > 0]$:	$\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$,	$\xi \propto (t_{\mathrm{s}} - t)^{-(q-1)\sigma}$
(ii) $[q > 1,\ \sigma < 0]$:	$\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$,	$\xi \sim \xi_{\mathrm{c}}$
(iii) $[-1 < q < 0\,,\ 0 < q < 1]$:	$\eta \sim \eta_{\mathrm{c}}$,	$\xi \sim \xi_{\mathrm{c}}$

→ We examine the behavior of each term of the gravitational field equations in the limit $t \to t_{\mathrm{s}}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained.

- For case (ii) $[q > 1,\ \sigma < 0]$, $\boxed{\xi_{\mathrm{c}} = 1}$
- For case (iii) $[-1 < q < 0\,,\ 0 < q < 1]$,

$$\boxed{f_{\mathrm{s}} \eta_{\mathrm{c}}^{\sigma-1} (6\sigma - \eta_{\mathrm{c}}) + \xi_{\mathrm{c}} - 1 = 0}$$

⇒ the leading term vanishes in both gravitational field equations.

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

⇒ Thus, the expression of the Hubble parameter can be a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field equations in the flat FLRW space-time.

⇓

This implies that there can exist the finite-time future singularities in non-local gravity.

B. Relations between the model parameters and the property of the finite-time future singularities

- $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$ $\longrightarrow$ f_{s} and σ characterize the theory of non-local gravity.
- $\boxed{H \sim \dfrac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{q}}}$ $\longrightarrow$ h_{s} , t_{s} and q specify the property of the finite-time future singularity.
- η_{c} and ξ_{c} determine a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field equations in the flat FLRW space-time.
- When $t \to t_{\mathrm{s}}$,

for $q > 1$, $a \to \infty$

for $-1 < q < 0$ and $0 < q < 1$, $a \to a_{\mathrm{s}}$

for $q > 0$, $H \to \infty$, $\rho_{\mathrm{eff}} = 3H^2/\kappa^2 \to \infty$

for $-1 < q < 0$, H asymptotically becomes finite and also ρ_{eff} asymptotically approaches a finite constant value ρ_{s} .

for $q > -1$, $\dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)} \to \infty$, $P_{\mathrm{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2 \to \infty$

- It is known that the finite-time future singularities can be classified in the following manner:

[Nojiri, Odintsov and Tsujikawa, Phys. Rev. D 71, 063004 (2005)]

	In the limit $t \to t_{\mathrm{s}}$,
Type I ("Big Rip"):	$a \to \infty\,, \quad \rho_{\mathrm{eff}} \to \infty\,, \quad \lvert P_{\mathrm{eff}} \rvert \to \infty$
	* The case in which ρ_{eff} and P_{eff} becomes finite values at $t = t_{\mathrm{s}}$ is also included.
Type II ("sudden"):	$a \to a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to \rho_{\mathrm{s}}\,, \quad \lvert P_{\mathrm{eff}} \rvert \to \infty$
Type III:	$a \to a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to \infty\,, \quad \lvert P_{\mathrm{eff}} \rvert \to \infty$
Type IV:	$a \to a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to 0\,, \quad \lvert P_{\mathrm{eff}} \rvert \to 0$
	* Higher derivatives of H diverge.
	* The case in which ρ_{eff} and/or $\lvert P_{\mathrm{eff}} \rvert$ asymptotically approach finite values is also included.

- The finite-time future singularities described by the expression of H in non-local gravity have the following properties:

$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}$$

For $q > 0$,	Type I (“Big Rip”)
For $-1 < q < 0$,	Type II (“sudden”)
For $q > -1$,	Type III

* Range and conditions for the value of parameters of $f(\eta)$, H , and η_{c} and ξ_{c} in order that the finite-time future singularities can exist.

Case	$f(\eta) = f_{\mathrm{s}} \eta^{\sigma}$	$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}}-t)^q}$	η_{c} , ξ_{c}
	$f_{\mathrm{s}} \neq 0$	$h_{\mathrm{s}} > 0$	$\eta_{\mathrm{c}} \neq 0$
	$\sigma \neq 0$	$q > -1$, $q \neq 0$	
(ii)	$\sigma < 0$	$q > 1$ [Type I (“Big Rip”) singularity]	$\xi_{\mathrm{c}} = 1$
(iii)	$f_{\mathrm{s}} \eta_{\mathrm{c}}^{\sigma-1} (6\sigma - \eta_{\mathrm{c}}) + \xi_{\mathrm{c}} - 1 = 0$	$0 < q < 1$ [Type III singularity]	
		$-1 < q < 0$ [Type II (“sudden”) singularity]	

IV. Effective equation of state for the universe and phantom-divide crossing

A. Cosmological evolution of the effective equation of state for the universe

- The effective equation of state for the universe : $\boxed{w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2}}$

$$\rho_{\mathrm{eff}} = \frac{3H^2}{\kappa^2}\,, \quad P_{\mathrm{eff}} = -\frac{2\dot{H} + 3H^2}{\kappa^2}$$

$\dot{H} < 0$: **The non-phantom (quintessence) phase**

$\longrightarrow \quad w_{\mathrm{eff}} > -1$

$\dot{H} = 0 \quad \longrightarrow \quad w_{\mathrm{eff}} = -1$ **Phantom crossing**

$\dot{H} > 0$: **The phantom phase**

$\longrightarrow \quad w_{\mathrm{eff}} < -1$

$\longrightarrow$ We examine the asymptotic behavior of w_{eff} in the limit $t \to t_{\mathrm{s}}$ by taking the leading term in terms of $(t_{\mathrm{s}} - t)$.

- For $q > 1$ [Type I ("Big Rip") singularity], w_{eff} evolves from the non-phantom phase or the phantom one and asymptotically approaches $w_{\mathrm{eff}} = -1$.

- For $0 < q < 1$ [Type III singularity], w_{eff} evolves from the non-phantom phase to the phantom one with realizing a crossing of the phantom divide or evolves in the phantom phase.

  $\Rightarrow$ The final stage is the eternal phantom phase.

- For $-1 < q < 0$ [Type II ("sudden") singularity], $w_{\mathrm{eff}} > 0$ at the final stage.

B. Estimation of the current value of the effective equation of state parameter for non-local gravity

[Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011)]

- The limit on a constant equation of state for dark energy in a flat universe has been estimated as

$$\boxed{w_{\mathrm{DE}} = -1.10 \pm 0.14 \, (68\% \, \mathrm{CL})}$$

by combining the data of Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations with the latest distance measurements from the baryon acoustic oscillations (BAO) in the distribution of galaxies and the Hubble constant measurement.

- For a time-dependent equation of state for dark energy, by using a linear form $w_{\mathrm{DE}}(a) = w_{\mathrm{DE}\,0} + w_{\mathrm{DE}\,a}\,(1-a)$, constraints on $w_{\mathrm{DE}\,0}$ and $w_{\mathrm{DE}\,a}$ have been found as

$w_{\mathrm{DE}\,0}$: Current value of w_{DE}

$w_{\mathrm{DE}\,a}$: Derivative of w_{DE}

$$\boxed{\begin{aligned} w_{\mathrm{DE}\,0} &= -0.93 \pm 0.13 \,, \\ w_{\mathrm{DE}\,a} &= -0.41^{+0.72}_{-0.71} \, (68\% \, \mathrm{CL}) \end{aligned}}$$

from the combination of the WMAP data with the BAO data, the Hubble constant measurement and the high-redshift SNe Ia data.

→ We estimate the present value of $w_{\rm eff}$.

* We regard $w_{\rm eff} \approx w_{\rm DE}$ at the present time because the energy density of dark energy is dominant over that of non-relativistic matter at the present time.

- For case (ii) $[q > 1, \sigma < 0]$,

$\sigma = -1$

$q = 2$

$h_{\rm s} = 1\,[{\rm GeV}]^{-1}$

$t_{\rm s} = 2t_{\rm p}$

$H_{\rm p} = 2.1h \times 10^{-42}{\rm GeV}$

: Current value of H , $h = 0.7$

[Freedman *et al*. [HST Collaboration], Astrophys. J. 553, 47 (2001)]

$$\boxed{\begin{array}{l} f_{\rm s} = -3.0 \times 10^{-43} \\ \underline{w_{\rm eff} = -1.10} \end{array}} \qquad \boxed{\begin{array}{l} f_{\rm s} = -2.1 \times 10^{-43} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

$t_{\rm p}$: The present time

$h_{\rm s}$ has the dimension of $[{\rm Mass}]^{q-1}$.

- For $0 < q < 1$,

$q = 1/2$

$h_{\rm s} = 1\,[{\rm GeV}]^{1/2}$

$\eta_{\rm c} = 1$

$t_{\rm s} = 2t_{\rm p}$

$$\boxed{\begin{array}{l} f_{\rm s} = 7.9 \times 10^{-2} \\ \underline{w_{\rm eff} = -1.10} \end{array}} \qquad \boxed{\begin{array}{l} f_{\rm s} = 6.6 \times 10^{-2} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

- For $-1 < q < 0$, $w_{\rm eff} > 0$.

$\Rightarrow$ **In our models, $w_{\rm eff}$ can have the present observed value of $w_{\rm DE}$.**

C. Cosmological consequences of adding an R^2 term

→ We explore whether the addition of an R^2 term removes the finite-time future singularities in non-local gravity.

< Action >

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\Box^{-1}R)\right)\underline{+uR^2}-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\left(Q;g\right)\right\}$$

- Gravitational field equations in the flat FLRW background: $u(\neq 0)$

$$0=-3H^2\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)\underline{+\Theta}+\Lambda+\kappa^2\rho_{\mathrm{m}}$$

$$0=\left(2\dot{H}+3H^2\right)\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)\left(f(\eta)-\xi\right)\underline{+\Xi}-\Lambda+\kappa^2P_{\mathrm{m}}$$

- In the limit $t\longrightarrow t_{\mathrm{s}}$,

→ $$\underline{\Theta}\sim 18u\left[-6h_{\mathrm{s}}^2q\left(t_{\mathrm{s}}-t\right)^{-(3q+1)}+h_{\mathrm{s}}^2q^2\left(t_{\mathrm{s}}-t\right)^{-2(q+1)}-2h_{\mathrm{s}}^2q\left(q+1\right)\left(t_{\mathrm{s}}-t\right)^{-2(q+1)}\right]$$

→ $$\underline{\Xi}\sim 6u\left[9h_{\mathrm{s}}^2q^2\left(t_{\mathrm{s}}-t\right)^{-2(q+1)}+18h_{\mathrm{s}}^3q\left(t_{\mathrm{s}}-t\right)^{-(3q+1)}\right.$$
$$\left.+2h_{\mathrm{s}}q\left(q+1\right)\left(q+2\right)\left(t_{\mathrm{s}}-t\right)^{-(q+3)}+12h_{\mathrm{s}}^2q\left(q+1\right)\left(t_{\mathrm{s}}-t\right)^{-2(q+1)}\right]$$

The leading terms do not vanish. ⇒ **The additional R^2 term can remove the finite-time future singularity.**

V. Summary

- **We have studied de Sitter solutions in non-local gravity.**

  ⇨ **We have explored a condition to avoid a ghost and presented a screening scenario for a cosmological constant in de Sitter solutions.**

- **We have explicitly shown that three types of the finite-time future singularities (Type I, II and III) can occur in non-local gravity and examined their properties.**

- **We have investigated the behavior of the effective equation of state for the universe when the finite-time future singularities occur.**

  ⇨ **We have demonstrated that the addition of an R^2 term can remove the finite-time future singularities in non-local gravity.**

< Further results and remarks >

- **We have also studied de Sitter solutions in non-local gravity with Lagrange constraint multiplier.**

- **It has also been suggested that the addition of an R^2 term in the framework of non-local gravity might realize unification of inflation in the early universe with the cosmic acceleration in the late time.**

Backup slides

- In the presence of matter with $w_{\rm m} \neq 0$, for $\Lambda = 0$, we may have a de Sitter solution $H = H_0$ even if $f(\eta)$ is given by

$$f(\eta) = f_0 \mathrm{e}^{\eta/2} + f_1 \mathrm{e}^{3(w_{\rm m}+1)\eta/4}$$

⇨ Therefore, the following solution exists:

$$\eta = -4H_0 t$$

$$\xi = 1 - 3f_0 \mathrm{e}^{-2H_0 t} + \frac{f_1}{w_{\rm m}} \mathrm{e}^{-3(w_{\rm m}+1)H_0 t}$$

$$\rho_{\rm m} = -\frac{3(3w_{\rm m}+1)H_0^2 f_1}{\kappa^2} \mathrm{e}^{-3(1+w_{\rm m})H_0 t}$$

We may introduce a new field $\chi = \int^{\eta} \sqrt{f'(\eta)} d\eta$.

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left\{ \frac{1}{2\kappa^2} \left[R^{(\mathrm{E})} - 6\nabla^{\mu}\phi\nabla_{\mu}\phi - \frac{2}{\sqrt{f'}}\nabla^{\mu}\phi\nabla_{\mu}\chi - \mathrm{e}^{2\phi}\nabla^{\mu}\chi\nabla_{\mu}\chi - 2\mathrm{e}^{4\phi}\Lambda \right] + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q; \mathrm{e}^{2\phi}g^{(\mathrm{E})}\right) \right\}$$

$$f'(\eta) = f'\left(\eta\left(\chi\right)\right)$$

C. Condition to be free of ghost

- To examine the ghost-free condition, we make a conformal transformation to the Einstein frame:

A superscription (E) represents quantities in the Einstein frame.

$$g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})}$$

$$\Omega^2 = \frac{1}{1+f(\eta)-\xi}$$

$$R = \frac{1}{\Omega^2}\left[R^{(\mathrm{E})} - 6\left(\Box \ln\Omega + g^{(\mathrm{E})\,\mu\nu}\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega\right)\right]$$

$$S = \int d^4x\sqrt{-g^{(\mathrm{E})}}\left\{\frac{1}{2\kappa^2}\left[R^{(\mathrm{E})} - 6\left(\underline{\Box \ln\Omega} + g^{(\mathrm{E})\,\mu\nu}\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega\right) - \Omega^2 g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\eta - 2\Omega^4\Lambda\right]\right.$$
$$\left. +\Omega^4 \mathcal{L}_{\mathrm{matter}}\left(Q;\Omega^2 g^{(\mathrm{E})}\right)\right\}$$

The $\Box \ln\Omega$ term may be discarded because it is a total divergence.

$$S = \int d^4x\sqrt{-g^{(\mathrm{E})}}\left[\frac{1}{2\kappa^2}\left(R^{(\mathrm{E})} - 6g^{(\mathrm{E})\,\mu\nu}\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega - \Omega^2 g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\eta - 2\Omega^4\Lambda\right)\right.$$
$$\left. +\Omega^4 \mathcal{L}_{\mathrm{matter}}\left(Q;\Omega^2 g^{(\mathrm{E})}\right)\right]$$
$$= \int d^4x\sqrt{-g^{(\mathrm{E})}}\left[\frac{1}{2\kappa^2}\left(R^{(\mathrm{E})} - 6g^{(\mathrm{E})\,\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \mathrm{e}^{2\phi} g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\eta - 2\mathrm{e}^{4\phi}\Lambda\right)\right.$$
$$\left. +\mathrm{e}^{4\phi} \mathcal{L}_{\mathrm{matter}}\left(Q;\mathrm{e}^{2\phi} g^{(\mathrm{E})}\right)\right]$$

$$\phi = \ln\Omega = -\frac{1}{2}\ln\left(1+f(\eta)-\xi\right)$$

- Instead of η and ξ, we may regard ϕ and η to be independent fields.

$$\longleftarrow \quad \xi = -\mathrm{e}^{-2\phi} + (1 + f(\eta))$$

$$\boxed{S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left\{ \frac{1}{2\kappa^2} \left[R^{(\mathrm{E})} - 6\nabla^\mu\phi\nabla_\mu\phi - 2\nabla^\mu\phi\nabla_\mu\eta - \mathrm{e}^{2\phi} f'(\eta)\nabla^\mu\eta\nabla_\mu\eta - 2\mathrm{e}^{4\phi}\Lambda \right] + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q; \mathrm{e}^{2\phi} g^{(\mathrm{E})}\right) \right\}}$$

- In order to avoid a ghost, the determinant of the kinetic term must be positive, which means

[Nojiri, Odintsov, Sasaki and Zhang, Phys. Lett. B <u>696</u>, 278 (2011)]

$$\det \begin{vmatrix} 6 & 1 \\ 1 & \mathrm{e}^{2\phi} f'(\eta) \end{vmatrix} = 6\mathrm{e}^{2\phi} f'(\eta) - 1 > 0$$

$\longrightarrow$ This condition is assumed to be satisfied. In particular, $f'(\eta) > 0$ is a necessary condition.

$$\boxed{f'(\eta) > \frac{1}{6\mathrm{e}^{2\phi}} = \frac{1 + f(\eta) - \xi}{6} > 0}$$

: The ghost-free condition

$\longrightarrow$ The phantom phase as well as the non-phantom phase can be realized in the Einstein frame in the context of non-local gravity.

If $f_0 = 2 \longrightarrow \quad \phi_0 = 3/\left(4 + 9\sqrt{2}\right), \quad h_0 = \pm 3/\left(2\sqrt{2} + 9\right)$

- We remark that since we now consider the case in which the contribution of matter is absent, w_{eff} is equivalent to the equation of state for dark energy.

$\longrightarrow$ This is because ρ_{eff} and P_{eff} correspond to ρ_{tot} and P_{tot}, respectively, and thus w_{eff} can be expressed as $w_{\text{eff}} = P_{\text{tot}}/\rho_{\text{tot}}$.

ρ_{tot} and P_{tot} : The total energy density and pressure of the universe, respectively.

- In the case $\rho_m = \Lambda = 0$, $\quad \dfrac{f_0}{\beta} > \dfrac{1}{6}$: **The ghost-free condition**

$$\Rightarrow \quad \frac{2f_0(1-f_0)}{3} > \frac{1}{6} \quad \leftrightarrow \quad f_0^2 - f_0 + \frac{1}{4} < 0 \quad \leftrightarrow \quad \left(f_0 - \frac{1}{2}\right)^2 < 0$$

This cannot be satisfied. Thus the solution also contains a ghost.

D. Cosmology in the Einstein frame

- We assume the FLRW metric, and consider the case in which the contribution of matter is negligible.
- The variation of the action in terms of ϕ and η leads to

$$0 = 12\left(\ddot{\phi} + 3H\dot{\phi}\right) + 2\left(\ddot{\eta} + 3H\dot{\eta}\right) - 2\mathrm{e}^{2\phi} f'(\eta)\dot{\eta}^2 + 8\mathrm{e}^{4\phi}\Lambda$$

$$0 = 2\left(\ddot{\phi} + 3H\dot{\phi}\right) + 2\left(\frac{d}{dt} + 3H\right)\left(\mathrm{e}^{2\phi} f'(\eta)\dot{\eta}\right) - e^{2\phi} f''(\eta)\dot{\eta}^2$$

- The first FLRW equation: $3H^2 = 3\dot{\phi}^2 - \dot{\phi}\dot{\eta} - \frac{\mathrm{e}^{2\phi}}{2} f'(\eta)\dot{\eta}^2 + \mathrm{e}^{4\phi}\Lambda$
- We investigate the case that $\Lambda = 0\,, \quad f'(\eta) = \frac{f_0}{\beta}\mathrm{e}^{\frac{\eta}{\beta}}$.
- We suppose $H = \frac{h_0}{t}\,, \quad \phi = \phi_0 \ln\frac{t}{t_0}\,, \quad \eta = -2\beta\phi_0 \ln\frac{t}{t_0}$.

$$a = a_0 t^{h_0}$$

$$\begin{cases} 0 = \left[\left(-1 + 3h_0\right)\left(12 - 4\beta\right) - 8f_0\phi_0\beta\right]\phi_0 \\ 0 = \left[\left(-1 + 3h_0\right)\left(2 - 4f_0\right) - 4f_0\phi_0\right]\phi_0 \\ 0 = -3h_0^2 + \left(3 + 2\beta - 2f_0\beta\right)\phi_0^2 \end{cases}$$

- There exists a flat solution where $h_0 = \phi_0 = 0$.

$f_0 \neq 1$

$\phi_0 \neq 0$

- If $\phi_0 \neq 0$, $2\beta = \dfrac{3}{1-f_0}$, $\phi_0 = \dfrac{(-1+3h_0)(2-4f_0)}{4f_0}$.

$\longrightarrow h_0 \neq 1/3$

$f_0 \neq 1/2$

$$\Rightarrow h_0 = \pm\sqrt{2}\phi_0 = \pm\frac{\sqrt{2}(2f_0-1)}{2f_0 \pm 3\sqrt{2}(2f_0-1)}$$

$$\phi_0 = \frac{(2f_0-1)}{2f_0 \pm 3\sqrt{2}(2f_0-1)}$$

- The effective equation of state for the universe :

$$\boxed{w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{2}{3h_0}}$$

$$\rho_{\mathrm{eff}} = \frac{3H^2}{\kappa^2}, \quad P_{\mathrm{eff}} = -\frac{2\dot{H}+3H^2}{\kappa^2}$$

$\dot{H} = -h_0/t^2 < 0$: The non-phantom (quintessence) phase

$\longrightarrow h_0 > 0$ $\qquad w_{\mathrm{eff}} > -1$

$\dot{H} = -h_0/t^2 > 0$: The phantom phase

$\longrightarrow h_0 < 0$ $\qquad w_{\mathrm{eff}} < -1$

→ The phantom phase as well as the non-phantom phase can be realized in the Einstein frame in the context of non-local gravity.

If $f_0 = 2 \longrightarrow \phi_0 = 3/\left(4 + 9\sqrt{2}\right)$, $h_0 = \pm 3/\left(2\sqrt{2} + 9\right)$

- In the case $\rho_m = \Lambda = 0$, $\dfrac{f_0}{\beta} > \dfrac{1}{6}$ **:** **The ghost-free condition**

$$\Rightarrow \quad \frac{2 f_0 (1 - f_0)}{3} > \frac{1}{6} \quad \leftrightarrow \quad f_0^2 - f_0 + \frac{1}{4} < 0 \quad \leftrightarrow \quad \left(f_0 - \frac{1}{2}\right)^2 < 0$$

This cannot be satisfied. Thus, the solution also contains a ghost.

E. Addition of an R^2 term

$\longrightarrow$ We examine the influence of adding an R^2 term on the stability of non-local gravity in the Einstein frame.

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\Box^{-1}R)\right)+uR^2-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\left(Q;g\right)\right\}$$

$\longleftarrow$ We introduce another scalar field ζ . $u(\neq 0)$

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)\right)-\partial_\mu\xi\partial^\mu\eta-\xi R+u\left(2\zeta R-\zeta^2\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\right\}$$

By varying the action with respect to ζ , we have $\zeta = R$. Substituting this equation into the action in the second expression, the starting action is re-obtained.

- Gravitational field equations in the flat FLRW background:

$$0 = -3H^2\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)+\Theta+\Lambda+\kappa^2\rho_{\mathrm{m}}$$

$$0 = \left(2\dot{H}+3H^2\right)\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)\left(f(\eta)-\xi\right)+\Xi-\Lambda+\kappa^2P_{\mathrm{m}}$$

$$\Theta \equiv u\left(-6H^2R+\frac{1}{2}R^2-6H\dot{R}\right)=18u\left(-6H^2\dot{H}+\dot{H}^2-2H\ddot{H}\right)$$

$$\Xi \equiv u\left[2\left(2\dot{H}+3H^2\right)R-\frac{1}{2}R^2+2\ddot{R}+4H\dot{R}\right]=6u\left(9\dot{H}^2+18H^2\dot{H}+2\dddot{H}+12H\ddot{H}\right)$$

- The effective equation of state for the universe is given by

$$w_{\mathrm{eff}} = \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{\left(2\dot{H}+3H^2\right)(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)(f(\eta)-\xi)-\Lambda+\kappa^2 P_{\mathrm{m}}}{-3H^2(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)+\Lambda+\kappa^2\rho_{\mathrm{m}}}$$

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2}\left[-3H^2(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)+\Lambda+\kappa^2\rho_{\mathrm{m}}\right]$$

$$P_{\mathrm{eff}} = \frac{1}{\kappa^2}\left[\left(2\dot{H}+3H^2\right)(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)(f(\eta)-\xi)-\Lambda+\kappa^2 P_{\mathrm{m}}\right]$$

$\Downarrow$ If we add an R^2 term as in the action, ρ_{eff} and P_{eff} become

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2}\left[-3H^2(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)\underline{+\Theta}+\Lambda+\kappa^2\rho_{\mathrm{m}}\right]$$

$$P_{\mathrm{eff}} = \frac{1}{\kappa^2}\left[\left(2\dot{H}+3H^2\right)(f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)(f(\eta)-\xi)\right.$$
$$\left.\underline{+\Xi}-\Lambda+\kappa^2 P_{\mathrm{m}}\right]$$

- We examine the condition to avoid a ghost in the present case.

→ By following the same procedure in Sec. II C, we perform a conformal transformation to the Einstein frame.

$$\boxed{g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})}\,, \quad \Omega^2 = \frac{1}{1+f(\eta)-\xi+2u\zeta}}$$

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(\mathrm{E})} - 6 g^{(\mathrm{E})\,\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \mathrm{e}^{2\phi} g^{\mu\nu} \nabla_\mu \xi \nabla_\nu \eta - u \mathrm{e}^{4\phi} \zeta^2 - 2\mathrm{e}^{4\phi} \Lambda \right) + \mathrm{e}^{4\phi} \mathcal{L}_{\mathrm{matter}} \left(Q; \mathrm{e}^{2\phi} g^{(\mathrm{E})} \right) \right]$$

$$\phi = \ln \Omega = -(1/2) \ln \left(1 + f(\eta) - \xi + 2u\zeta \right)$$

← Substitution of $\xi = -\mathrm{e}^{-2\phi} + (1+f(\eta)) + 2u\zeta$

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(\mathrm{E})} - 6 \nabla^\mu \phi \nabla_\mu \phi - 2 \nabla^\mu \phi \nabla_\mu \eta - \mathrm{e}^{2\phi} f'(\eta) \nabla^\mu \eta \nabla_\mu \eta - 2u \mathrm{e}^{2\phi} \nabla^\mu \zeta \nabla_\mu \eta - u \mathrm{e}^{4\phi} \zeta^2 - 2\mathrm{e}^{4\phi} \Lambda \right) + \mathrm{e}^{4\phi} \mathcal{L}_{\mathrm{matter}} \left(Q; \mathrm{e}^{2\phi} g^{(\mathrm{E})} \right) \right]$$

→ The mass matrix is given by $M \equiv \begin{pmatrix} 6 & 1 & u\mathrm{e}^{2\phi} \\ 1 & \mathrm{e}^{2\phi} f'(\eta) & 0 \\ u\mathrm{e}^{2\phi} & 0 & 0 \end{pmatrix}$.

- **The necessary condition to avoid a ghost is that all the eigenvalues of the mass matrix M must be positive.**

$\longrightarrow$ The characteristic equation for M is given by

$$\det|M - yE| = \det\begin{vmatrix} 6-y & 1 & ue^{2\phi} \\ 1 & e^{2\phi}f'(\eta)-y & 0 \\ ue^{2\phi} & 0 & -y \end{vmatrix} = 0$$

y denotes an eigenvalue of M.

E : Unit matrix

$$\Rightarrow\; y^3 - Y_1 y^2 + Y_2 y - Y_3 = 0$$

$$Y_1 \equiv y_1 + y_2 + y_3 = e^{2\phi}f'(\eta) + 6$$

$$Y_2 \equiv y_2 y_3 + y_3 y_1 + y_1 y_2 = 6e^{2\phi}f'(\eta) - u^2 e^{4\phi} - 1$$

$$Y_3 \equiv y_1 y_2 y_3 = -u^2 e^{6\phi} f'(\eta)$$

$$\boxed{y_1 > 0, \quad y_2 > 0, \quad y_3 > 0}$$

: Three eigenvalues of M

$\Downarrow$

$$\boxed{\begin{array}{cc} Y_1 > 0, \quad Y_2 > 0, & Y_3 > 0 \\ (\text{i.e., } \mathrm{Tr}\, M > 0) & (\text{i.e., } \det|M| > 0) \end{array}}$$

If $f'(\eta) < 0$, $Y_3 > 0$

If $-6e^{-2\phi} < f'(\eta)$, $Y_1 > 0$

If $f'(\eta) < 0$, $Y_2 < 0$

All the three eigenvalues of M cannot be positive simultaneously. Thus, the necessary condition to avoid a ghost cannot be satisfied.

III. Finite-time future singularities in non-local gravity

A. Finite-time future singularities

- We examine whether there exists the finite-time future singularities in non-local gravity.

  → In the flat FLRW space-time, we analyze an asymptotic solution of the gravitational field equations in the limit of the time t_{s} when the finite-time future singularities appear.

- We consider the case in which the Hubble parameter is expressed as

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

h_{s} : Positive constant

q : Non-zero constant larger than -1 $(q > -1, q \neq 0)$

We only consider the period $0 < t < t_{\mathrm{s}}$.

When $t \to t_{\mathrm{s}}$,

for $q > 1$, $H \sim h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-q} \to \infty$

$\dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)} \to \infty$

$\Rightarrow R = 6\dot{H} + 12H^2 \to \infty$

for $-1 < q < 0$ and $0 < q < 1$, H is finite, but $\dot{H}$ becomes infinity and therefore R also diverges.

$$\longrightarrow \quad a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}}-t)^{-(q-1)}\right]$$

a_{S} : Constant

- By using $\ddot{\eta} + 3H\dot{\eta} = a^{-3} d\left(a^3 \dot{\eta}\right)/dt$ and $0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$,

$$\eta = -\int^t \frac{1}{a^3}\left(\int^{\bar{t}} R a^3 d\bar{t}\right) dt$$

- Taking the leading term in terms of $(t_{\mathrm{s}} - t)$, we obtain

for $q > 1$

$$\eta \sim \underline{-\frac{4h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}}-t)^{-(q-1)}} + \eta_{\mathrm{c}}$$

$\dot{H} \ll H^2$

$\rightarrow R \sim 12H^2$

Leading term

η_{c} : Integration constant

for $-1 < q < 0$

$0 < q < 1$

$$\eta \sim -\frac{6h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}}-t)^{-(q-1)} \underline{+ \eta_{\mathrm{c}}}$$

Leading term

$\dot{H} \gg H^2$

$\rightarrow R \sim 6\dot{H}$

Cf. If $q = 1$,

$$\eta \sim 6h_{\mathrm{s}}\left[(1+2h_{\mathrm{s}})/(1+3h_{\mathrm{s}})\right]\ln(t_{\mathrm{s}}-t) + \eta_{\mathrm{c}}$$

B. Analysis for $\eta_{\mathrm{c}} \neq 0$

We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$.

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$: Non-zero constants

By using $\ddot{\xi} + 3H\dot{\xi} = a^{-3} d\left(a^3 \dot{\xi}\right)/dt$ and $0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\eta)$,

$$\xi = \int^t \frac{1}{a^3} \left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} R a^3 d\bar{t} \right) dt$$

- Taking the leading term in terms of $(t_{\mathrm{s}} - t)$, we acquire

for $q > 1$

$$\xi \sim \underbrace{-f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1} \right)^{\sigma} (t_{\mathrm{s}} - t)^{-(q-1)\sigma}}_{\text{If } \sigma > 0} + \underbrace{\xi_{\mathrm{c}}}_{\text{If } \sigma < 0}$$

$R \sim 12H^2$ (for $q > 1$)

Leading term

ξ_{c} : Integration constant

for $-1 < q < 0$

$0 < q < 1$

$$\xi \sim \frac{6 f_{\mathrm{s}} h_{\mathrm{s}} \sigma \eta_{\mathrm{c}}^{\sigma-1}}{q-1} (t_{\mathrm{s}} - t)^{-(q-1)} + \underline{\xi_{\mathrm{c}}}$$

$R \sim 6\dot{H}$ (for $q < 1$)

Leading term

$\Longrightarrow$ Thus, there are three cases.

(i) $[q > 1,\ \sigma > 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$, $\xi \propto (t_{\mathrm{s}} - t)^{-(q-1)\sigma}$

(ii) $[q > 1,\ \sigma < 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$, $\xi \sim \xi_{\mathrm{c}}$

(iii) $[-1 < q < 0\,,\ 0 < q < 1]$: $\eta \sim \eta_{\mathrm{c}}$, $\xi \sim \xi_{\mathrm{c}}$

$\longrightarrow$ We examine the behavior of each term on the right-hand side (r.h.s.) of the gravitational field equations in the limit $t \to t_{\mathrm{s}}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained.

- We analyze the case of $\eta_{\mathrm{c}} \neq 0$ and that of $\eta_{\mathrm{c}} = 0$ separately.
- When $t \to t_{\mathrm{s}}$, Λ, ρ_{m} and P_{m} can be neglected because these values are finite.

IV. Effective equation of state for the universe and phantom-divide crossing

A. Cosmological evolution of the effective equation of state for the universe

- We examine the asymptotic behavior of w_{eff} in the limit $t \to t_{\mathrm{s}}$ by taking the leading term in terms of $(t_{\mathrm{s}} - t)$.

Case (ii) $[q > 1, \ \sigma < 0]$ [Type I ("Big Rip") singularity]

$$w_{\mathrm{eff}} \sim -1 + I(t) \sim -1$$

$$I(t) = -8\sigma f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1} \right)^{\sigma-1} (t_{\mathrm{s}} - t)^{(q-1)(1-\sigma)}$$

: Deviation w_{eff} of from -1

If $(-)^{\sigma-1} f_{\mathrm{s}} < 0$, $I(t)$ evolves from $I(t) > 0$ to $I(t) = 0$.

$$\Longrightarrow \ w_{\mathrm{eff}} > -1 \longrightarrow w_{\mathrm{eff}} = -1$$

If $(-)^{\sigma-1} f_{\mathrm{s}} > 0$, $I(t)$ evolves from $I(t) < 0$ to $I(t) = 0$.

$$\Longrightarrow \ w_{\mathrm{eff}} < -1 \longrightarrow w_{\mathrm{eff}} = -1$$

Case (iii) $[-1 < q < 0\,,\, 0 < q < 1]$

For $0 < q < 1$ [Type III singularity]

$$w_{\mathrm{eff}} \sim -1 + I(t) \sim -\frac{2q}{3h_{\mathrm{s}}}\left(t_{\mathrm{s}} - t\right)^{q-1}$$

$$I(t) = I_0 - \frac{2q}{3h_{\mathrm{s}}}\left(t_{\mathrm{s}} - t\right)^{q-1} \qquad I_0 = 1 + 2f_{\mathrm{s}}\sigma\eta_{\mathrm{c}}^{\sigma-2}\left[6\left(\sigma-1\right) - 7\eta_{\mathrm{c}}\right]$$

: Constant part of $I(t)$

If $I_0 > 0$, a crossing of the phantom divide from the non-phantom phase to the phantom one can occur because the sign of the second term in $I(t)$ is negative and the absolute value of the amplitude becomes very large.

If $I_0 < 0$, $I(t)$ always evolves in the phantom phase ($w_{\mathrm{eff}} < -1$).

$\longrightarrow$ Thus the final stage is the phantom phase and it is eternal.

For $-1 < q < 0$ [Type II (“sudden”) singularity]

$$w_{\mathrm{eff}} \sim -2h_{\mathrm{s}}q\left(t_{\mathrm{s}} - t\right)^{-(q+1)} / \left(\Lambda + \kappa^2\rho_{\mathrm{m}}\right)$$

If we consider $\Lambda > 0$, we have $w_{\mathrm{eff}} > 0$.

- It is known that in $F(R)$ gravity, the addition of an R^2 term could cure the finite-time future singularities.
- It has been suggested that in the framework of non-local gravity combined with an R^2 term, inflation in the early universe as well as the cosmic acceleration in the late time could be realized.
- The additional R^2 term leads to inflation and the late-time cosmic acceleration occurs due to the term of non-local gravity $Rf(\Box^{-1}R)$ in the action.

- **We have shown that the late-time accelerating universe may be effectively the quintessence, cosmological constant or phantom-like phases.**

- **We have also demonstrated that there is a case with realizing a crossing of the phantom divide from the non-phantom (quintessence) phase to the phantom one in the limit of the appearance of a finite-time future singularity.**

$\longrightarrow$ The estimation of the current value of the effective equation of state parameter for the universe which could be phantomic one around -1 shows that its observed value could be easily realized by the appropriate choice of non-local gravity parameters.

- **We have considered the cosmological consequences of adding an R^2 term.**

Non-local gravity with Lagrange constraint multiplier

III. Non-local gravity with Lagrange constraint multiplier

- We generalize non-local gravity by introducing Lagrange constraint multiplier and examine a de Sitter solution in non-local gravity with Lagrange constraint multiplier. **No. L2**

< The constrained action for a scalar field ψ >

$$S_\psi = \int d^4x\sqrt{-g}\left[-\lambda\left(\frac{1}{2}\partial_\mu\psi\partial^\mu\psi + U(\psi)\right)\right]$$

λ : Lagrange multiplier field

$\partial_\mu\psi$: Time-like vector

Constraint: $\boxed{(1/2)\,\partial_\mu\psi\partial^\mu\psi + U(\psi) = 0} \longrightarrow (1/2)\,(d\psi/dt)^2 = U(\psi)$

- We choose $U(\psi) = U_0$. $\longrightarrow \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + U_0 = 0$

U_0 : Constant

$\longrightarrow$ Under the constraint, we define

$n,\ \alpha,\ \gamma$: Constants

$$R^{(2n+2)} \equiv R - 2\kappa^2\alpha\left[\left(\partial^\mu\psi\partial^\nu\psi\nabla_\mu\nabla_\nu + 2U_0\nabla^\rho\nabla_\rho\right)^n\left(\partial^\mu\psi\partial^\nu\psi R_{\mu\nu} + U_0 R\right)\right]^2$$

$$R^{(2n+3)} \equiv R - 2\kappa^2\alpha\left[\left(\partial^\mu\psi\partial^\nu\psi\nabla_\mu\nabla_\nu + 2U_0\nabla^\rho\nabla_\rho\right)^n\left(\partial^\mu\psi\partial^\nu\psi R_{\mu\nu} + U_0 R\right)\right]$$
$$\times\left[\left(\partial^\mu\psi\partial^\nu\psi\nabla_\mu\nabla_\nu + 2U_0\nabla^\rho\nabla_\rho\right)^{n+1}\left(\partial^\mu\psi\partial^\nu\psi R_{\mu\nu} + U_0 R\right)\right]$$

$$\Box^{(n)} \equiv \Box + \gamma\left(\partial^\mu\psi\partial^\nu\psi\nabla_\mu\nabla_\nu + 2U_0\nabla^\rho\nabla_\rho\right)^n$$

< Non-local action >

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R^{(m)}\left(1+f\left(\left(\square^n\right)^{-1}R^{(m)}\right)\right)-2\Lambda\right]-\lambda\left(\frac{1}{2}\partial_\mu\psi\partial^\mu\psi+U_0\right)+\mathcal{L}_{\mathrm{matter}}\right\}$$

$\Downarrow$ ← Introducing two scalar fields η and ξ

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R^{(m)}\left(1+f(\eta)\right)+\xi\left(\square^{(n)}\eta-R^{(m)}\right)-2\Lambda\right]-\lambda\left(\frac{1}{2}\partial_\mu\psi\partial^\mu\psi+U_0\right)+\mathcal{L}_{\mathrm{matter}}\right\}$$

→ We assume the following form of the metric:

n can be even or odd integer.

$$\boxed{ds^2=-\mathrm{e}^{2b(t)}dt^2+a^2(t)\sum_{i=1,2,3}\left(dx^i\right)^2}$$, and ψ only depends on time.

$$\Rightarrow\quad \begin{aligned}&\partial^\mu\psi\partial^\nu\psi R_{\mu\nu}+U_0R=6U_0\mathrm{e}^{-2b}H^2\\&\partial^\mu\psi\partial^\nu\psi\nabla_\mu\nabla_\nu+2U_0\nabla^\rho\nabla_\rho=-6U_0\mathrm{e}^{-2b}H\partial_t\\&R=\mathrm{e}^{-2b}\left(6\dot{H}+12H^2-6\dot{b}H\right)\end{aligned}$$

- We suppose η and ξ only depend on t .
- We examine most simple but non-trivial case that $m=2$.

(i.e., $n=0$, $\gamma=0$)

$$\longrightarrow R^{(2)}=R-2\kappa^2\alpha\left(\partial^\mu\psi\partial^\nu\psi R_{\mu\nu}+U_0R\right)^2$$

- The variation of the action with respect to η gives

$$0 = \Box\xi + f'(\eta)R^{(2)} \Rightarrow 0 = \left(6\dot{H} + 12H^2 - 72\kappa^2\alpha U_0^2 H^4\right) f'(\eta) - \ddot{\xi} - 3H\dot{\xi}$$

in the above background, after putting $b = 0$.

- The variation of the action with respect to ξ leads to

$$\Box\eta = R^{(2)} \Rightarrow 0 = 6\dot{H} + 12H^2 - 72\kappa^2\alpha U_0^2 H^4 + \ddot{\eta} + 3H\dot{\eta}$$

in the above background.

- For $m = 2$, the action is expressed as

$$S = \int d^4x a^3 \left(\frac{1}{2\kappa^2}\left\{\left[\mathrm{e}^{-b}\left(6\dot{H} + 12H^2 - 6\dot{b}H\right) - 72\kappa^2\alpha U_0^2 \mathrm{e}^{-3b} H^4\right](1 + f(\eta) - \xi)\right.\right.$$
$$\left.\left. + \mathrm{e}^{-b}\dot{\xi}\dot{\eta} - 2\mathrm{e}^{b}\Lambda\right\} - \lambda\left(-\frac{\mathrm{e}^{-b}}{2}\dot{\psi}^2 + \mathrm{e}^{b}U_0\right) + \mathcal{L}_{\mathrm{matter}}\right)$$

→ By the variation of the action with respect to b, we obtain the equation corresponding to the first FLRW equation:

* We have used $(1/2)\left(d\psi/dt\right)^2 = U_0$

↓

$$0 = -3\left(H^2 + 36\kappa^2\alpha U_0^2 H^4\right)(1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + 2\kappa^2\lambda U_0 + \kappa^2\rho_{\mathrm{m}}$$

→ The variation of the action with respect to a gives the equation corresponding to the second FLRW equation:

$$0 = \left(2\dot{H} + 3H^2 + 108\kappa^2\alpha U_0^2 H^4 + 144\kappa^2\alpha U_0^2 H^2 \dot{H}\right)(1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta}$$
$$+ \left(2H + 48\kappa^2\alpha U_0^2 H^3\right)\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + f''(\eta)\left(\dot{\eta}\right)^2 + f'(\eta)\ddot{\eta} - \ddot{\xi} - \Lambda + \kappa^2 P_{\mathrm{m}}$$

- We investigate if there could be a solution describing the de Sitter space. ⇒ We assume $\boxed{H = H_0}$.

→ $\eta = -4H_0\left(1 - 6\kappa^2\alpha U_0^2 H_0^2\right)t - \eta_0 \mathrm{e}^{-3H_0 t} + \eta_1$

* We take $\eta_0 = \eta_1 = 0$.

- We suppose $\boxed{f(\eta) = f_0 \mathrm{e}^{\frac{\eta}{\beta}}}$.

→ $\xi = -\dfrac{3f_0 H_0 t_{\mathrm{c}}}{1 + 3H_0 t_{\mathrm{c}}}\mathrm{e}^{\frac{t}{t_{\mathrm{c}}}} + \dfrac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1$, $\xi_0,\ \xi_1$: Constants of the integration

$$t_{\mathrm{c}} \equiv -\frac{\beta}{4H_0\left(1 - 6\kappa^2\alpha U_0^2 H_0^2\right)}$$

$$\xi_0 = 0\,, \quad \xi_1 = -1 + \frac{\Lambda}{3\left(H_0^2 + 36\kappa^2\alpha U_0^2 H_0^4\right)}$$

$$0 = 18\left(1 + 36\kappa^2\alpha U_0^2 H_0^2\right) H_0^3 \underline{t_c^3} + 3\left(7 + 120\kappa^2\alpha U_0^2 H_0^2\right) H_0^2 \underline{t_c^2} + 8\left(1 + 6\kappa^2\alpha U_0^2 H_0^2\right) H_0 \underline{t_c} + 1$$

: Cubic algebraic equation with respect to t_c

Hence, there is always a real solution of t_c.

→ We can find the value of β .

Consequently, if we choose β properly, there always appears the solution describing the de Sitter universe.

$$\underline{H_0^2 = \frac{\Lambda}{3\left(1 + \xi_1\right)}} - 36\kappa^2\alpha U_0^2 H_0^4$$

ξ_1 **can be a screening of the cosmological constant.**

* The last term is a correction coming from the covariant Hořava like model.

- The non-local action often appears as a quantum correction.

→ This equation implies again that the correction effectively changes the value of the cosmological constant.

- If $\xi \sim 0$ in the early universe, where $t \sim 0$, this equation means

$$\boxed{\xi_1 \sim -\frac{3 f_0 H_0 t_c}{1 + 3 H_0 t_c}}$$

* If H_0 corresponds to the value of the Hubble parameter in the present universe, the second term could be negligible.

⇨ If $-\frac{3 f_0 H_0 t_c}{1+3 H_0 t_c}$ is positive and very large, the effective cosmological constant in the present universe could be very small.

- We explore a condition to avoid the appearance of a ghost.

→ We make a conformal transformation to the Einstein frame:

$$g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})} \qquad R^{(2)} = \frac{1}{\Omega^2}\left[R^{(2,\,\mathrm{E})} - 6\left(\Box \ln \Omega + g^{(\mathrm{E})\,\mu\nu}\nabla_\mu \ln \Omega \nabla_\nu \ln \Omega\right)\right]$$

* We describe the Lagrangian for the part of the Lagrange multiplier field as

$$\mathcal{L}_\lambda\left(\Upsilon; g\right) = -\lambda\left(\frac{1}{2}\partial_\mu\psi\partial^\mu\psi + U_0\right)$$

$$S = \int d^4x\sqrt{-g^{(\mathrm{E})}}\left[\frac{1}{2\kappa^2}\left(R^{(2,\mathrm{E})} - 6\nabla^\mu\phi\nabla_\mu\phi - 2\nabla^\mu\phi\nabla_\mu\eta - \mathrm{e}^{2\phi}f'(\eta)\nabla^\mu\eta\nabla_\mu\eta - 2\mathrm{e}^{4\phi}\Lambda\right)\right.$$

$$\left. + \mathrm{e}^{4\phi}\mathcal{L}_\lambda\left(\Upsilon; \mathrm{e}^{2\phi}g^{(\mathrm{E})}\right) + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q; \mathrm{e}^{2\phi}g^{(\mathrm{E})}\right)\right]$$

$$\boxed{f'(\eta) > \frac{1}{6\mathrm{e}^{2\phi}} = \frac{1 + f(\eta) - \xi}{6} > 0}$$

: **The ghost-free condition**

→ The condition to avoid a ghost is described by the following form:

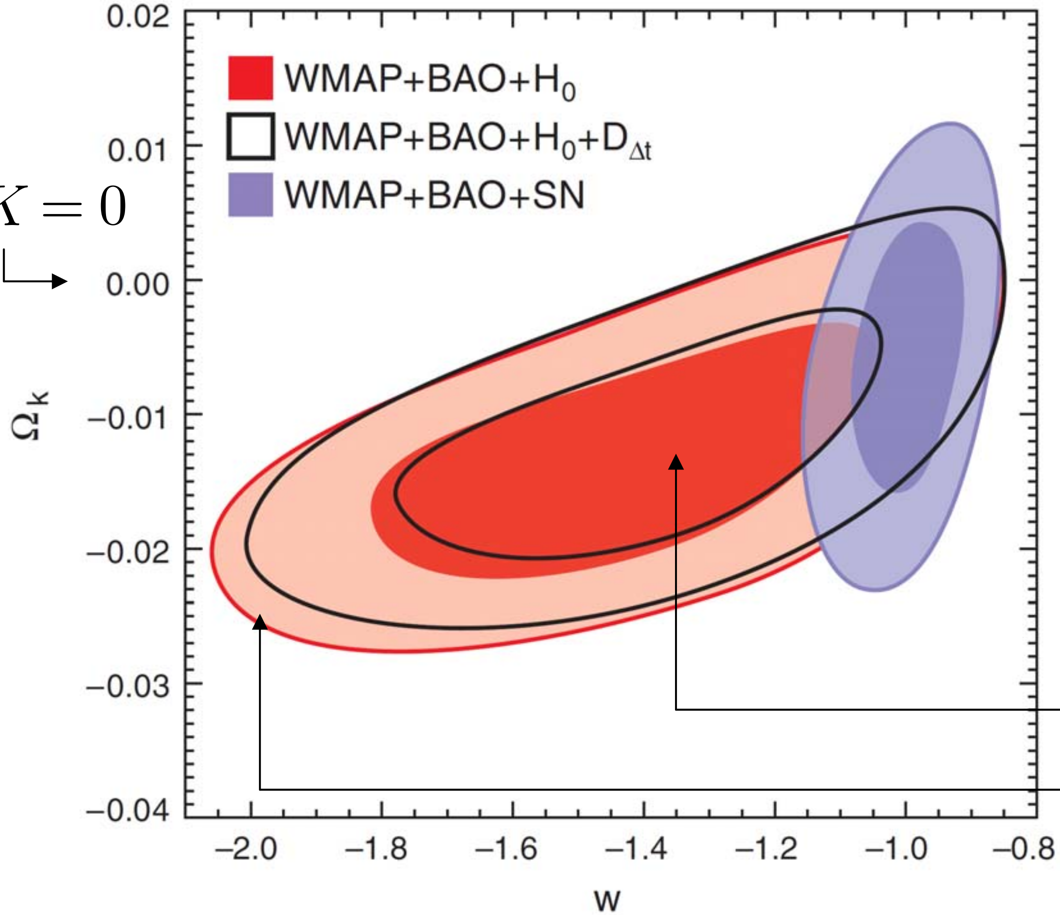
$$\frac{6}{\beta\left\{1 - \frac{3\beta}{4\left(1-6\kappa^2\alpha U_0^2 H_0^2\right)-3\beta} + \frac{\Lambda}{3f_0 H_0^2\left(1+36\kappa^2\alpha U_0^2 H_0^2\right)}\exp\left[\frac{4H_0\left(1-6\kappa^2\alpha U_0^2 H_0^2\right)}{\beta}t\right]\right\}} > 1$$

From a necessary condition $f'(\eta) > 0$, we find $f_0/\beta > 0$, which implies that the sign of f_0 is the same as that of β.

Appendix A

< 7-year WMAP data on the current value of w >

$K = 0$



$(68\%\ \mathrm{CL})$

$(95\%\ \mathrm{CL})$

From [E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

Hubble constant (H_0) measurement

Baryon acoustic oscillation (BAO)

: Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

$D_{\Delta t}$: Time delay distance

$$\Omega_K \equiv \frac{K}{(a_0 H_0)^2}$$

: Density parameter for the curvature

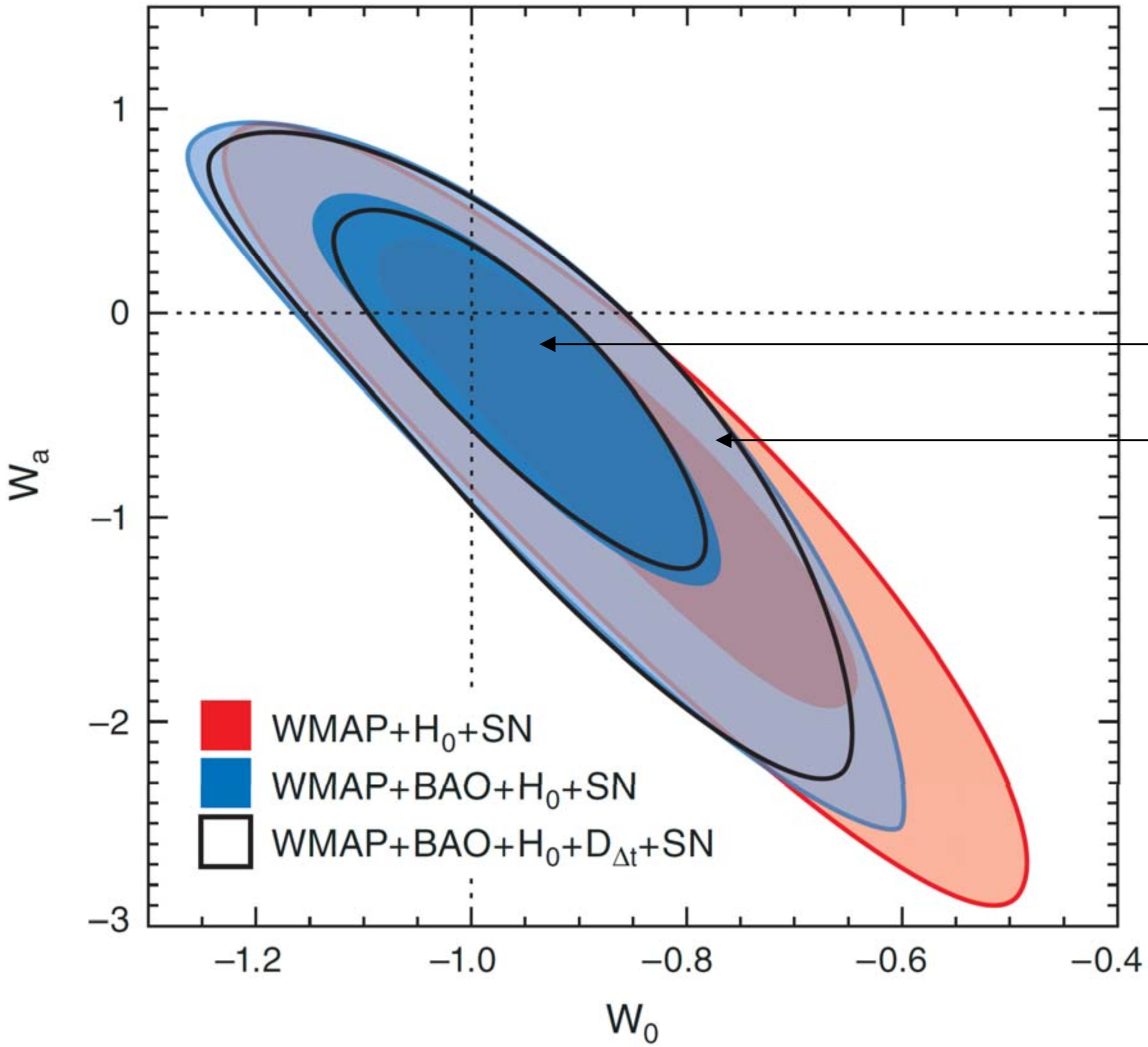
$K = 0$: Flat universe

- For the flat universe, constant w :

$$\boxed{w = -1.10 \pm 0.14\ (68\%\ \mathrm{CL})}$$

(From $WMAP$ $+\mathrm{BAO}+H_0$.)

Cf. $\Omega_\Lambda = 0.725 \pm 0.016\ \ (68\%\ \mathrm{CL})$



From [E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

(68% CL)

(95% CL)

Time-dependent w

$w(a) = w_0 + w_a(1 - a)$

$a = \frac{1}{1+z}$

w_0 : Current value of w

z : Redshift

(From $WMAP$+BAO+H_0+SN.)

- For the flat universe, a variable EoS :

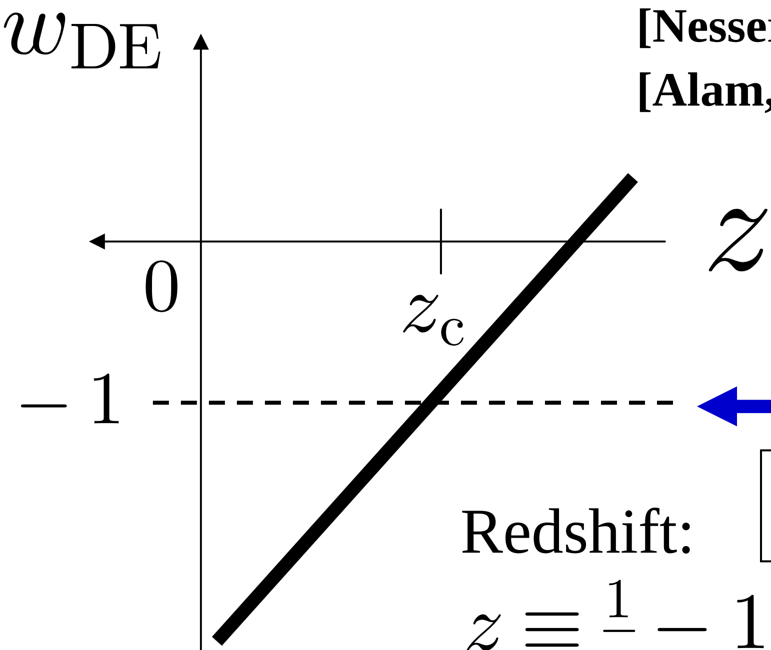
$$\boxed{w_0 = -0.93 \pm 0.13\,,\ w_a = -0.41^{+0.72}_{-0.71}\ \ (68\%\ \mathrm{CL})}$$

Appendix B

< Crossing of the phantom divide >

- Various observational data (SN, Cosmic microwave background radiation (CMB), BAO) imply that the effective EoS of dark energy w_{DE} may evolve from larger than -1 (non-phantom phase) to less than -1 (phantom phase). Namely, it crosses -1 (the crossing of the phantom divide).

[Alam, Sahni and Starobinsky, JCAP 0406, 008 (2004)]
[Nesseris and Perivolaropoulos, JCAP 0701, 018 (2007)]
[Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)]



Redshift: $z \equiv \frac{1}{a} - 1$

z_{c} : Red shift at the crossing of the phantom divide

(a) $w_{\mathrm{DE}} > -1$

Non-phantom phase

(b) $w_{\mathrm{DE}} = -1$

Crossing of the phantom divide

(c) $w_{\mathrm{DE}} < -1$

Phantom phase

< Effective equation of state for the universe >

$$\boxed{w_{\mathrm{eff}} \equiv -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = \frac{P_{\mathrm{tot}}}{\rho_{\mathrm{tot}}}}$$

$$\rho_{\mathrm{tot}} \equiv \rho_{\mathrm{DE}} + \rho_{\mathrm{m}} + \rho_{\mathrm{r}}$$

: Total energy density of the universe

$$P_{\mathrm{tot}} \equiv P_{\mathrm{DE}} + P_{\mathrm{m}} + P_{\mathrm{r}}$$

: Total pressure of the universe

$$\Longrightarrow \quad \boxed{w_{\mathrm{DE}} \approx w_{\mathrm{eff}}}$$

P_{DE} : Pressure of dark energy

P_{m} : Pressure of non-relativistic matter (cold dark matter and baryon)

P_{r} : Pressure of radiation

(a) $\dot{H} < 0 \Longrightarrow w_{\mathrm{eff}} > -1$ **Non-phantom phase**

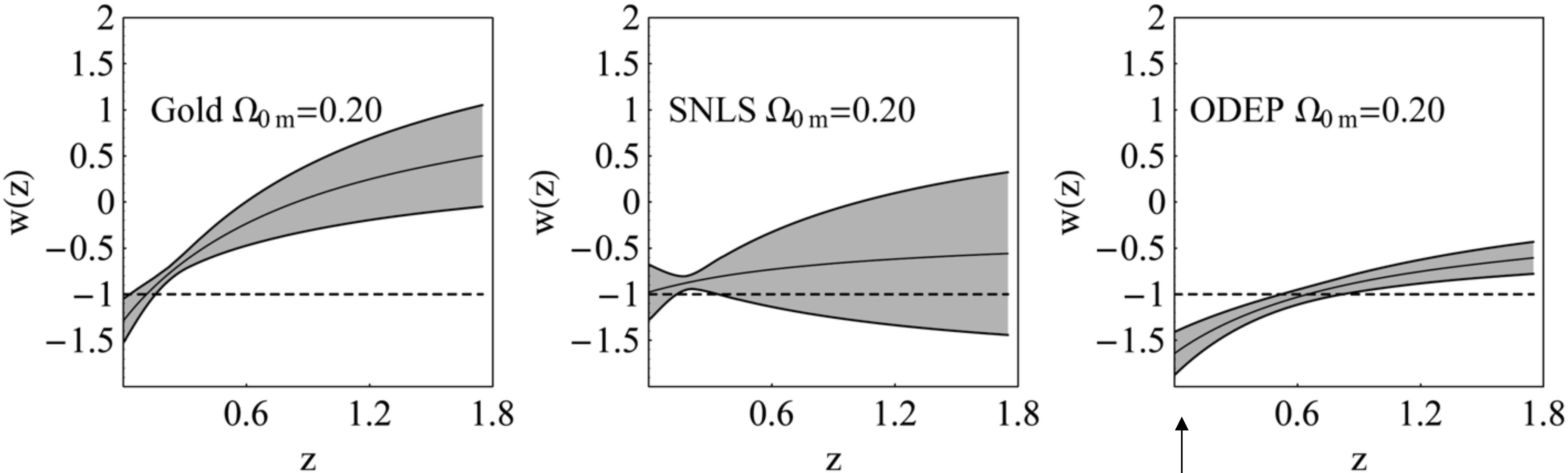
(b) $\dot{H} = 0 \Longrightarrow w_{\mathrm{eff}} = -1$ **Crossing of the phantom divide**

(c) $\dot{H} > 0 \Longrightarrow w_{\mathrm{eff}} < -1$ **Phantom phase**

< Data fitting of $w(z)$ (1) >

$$w(z) = w_0 + w_1 \frac{z}{1+z}$$

From [Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)].



SN gold data set

[Riess *et al*. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004)]

SNLS data set

[Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

Shaded region shows 1σ error.

Cosmic microwave background radiation (CMB) data

[Spergel *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007)]

+ SDSS baryon acoustic peak (BAO) data

[Eisenstein *et al*. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]

- For most observational probes (except the SNLS data), a low $\Omega_{0\mathrm{m}}$ prior $(0.2 < \Omega_{0\mathrm{m}} < 0.25)$ leads to an increased probability (mild trend) for the crossing of the phantom divide.

$\Omega_{0\mathrm{m}}$: Current density parameter of matter

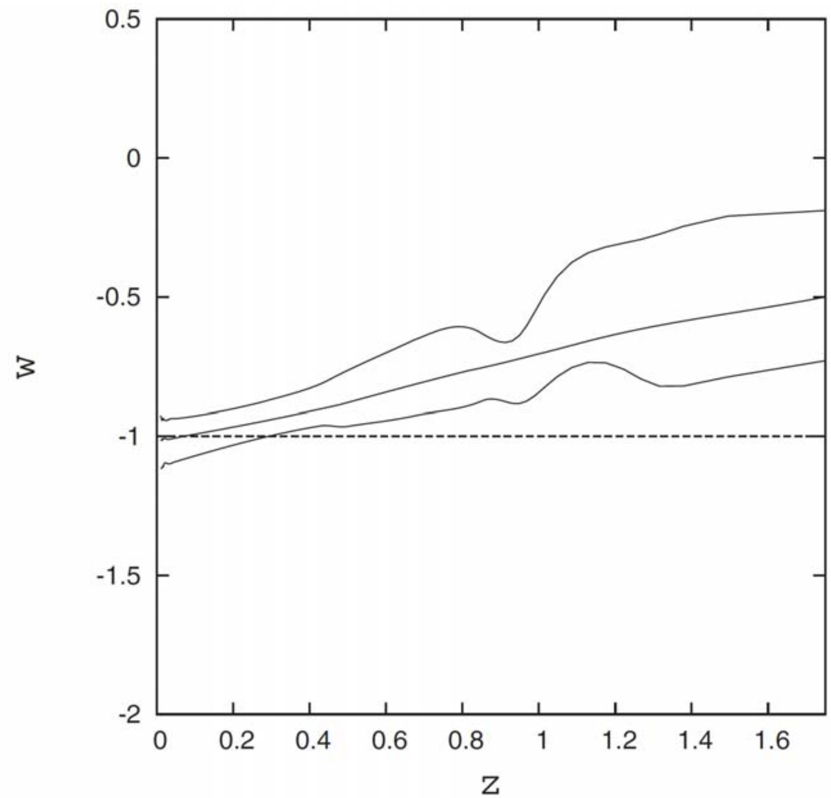
[Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)]

< Data fitting of $w(z)$ (2) >

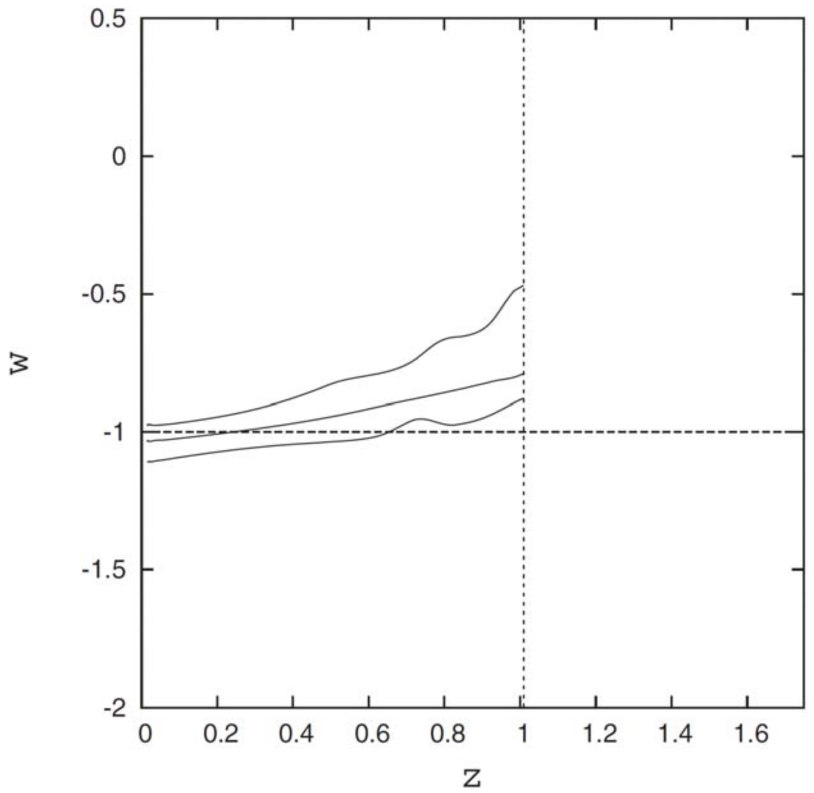
$$w(x) = \frac{(2x/3)\,\mathrm{d}\ln H/\mathrm{d}x - 1}{1 - (H_0/H)^2\Omega_{0m}x^3}$$

$x = 1 + z$

From [Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)].



SN gold data set+CMB+BAO



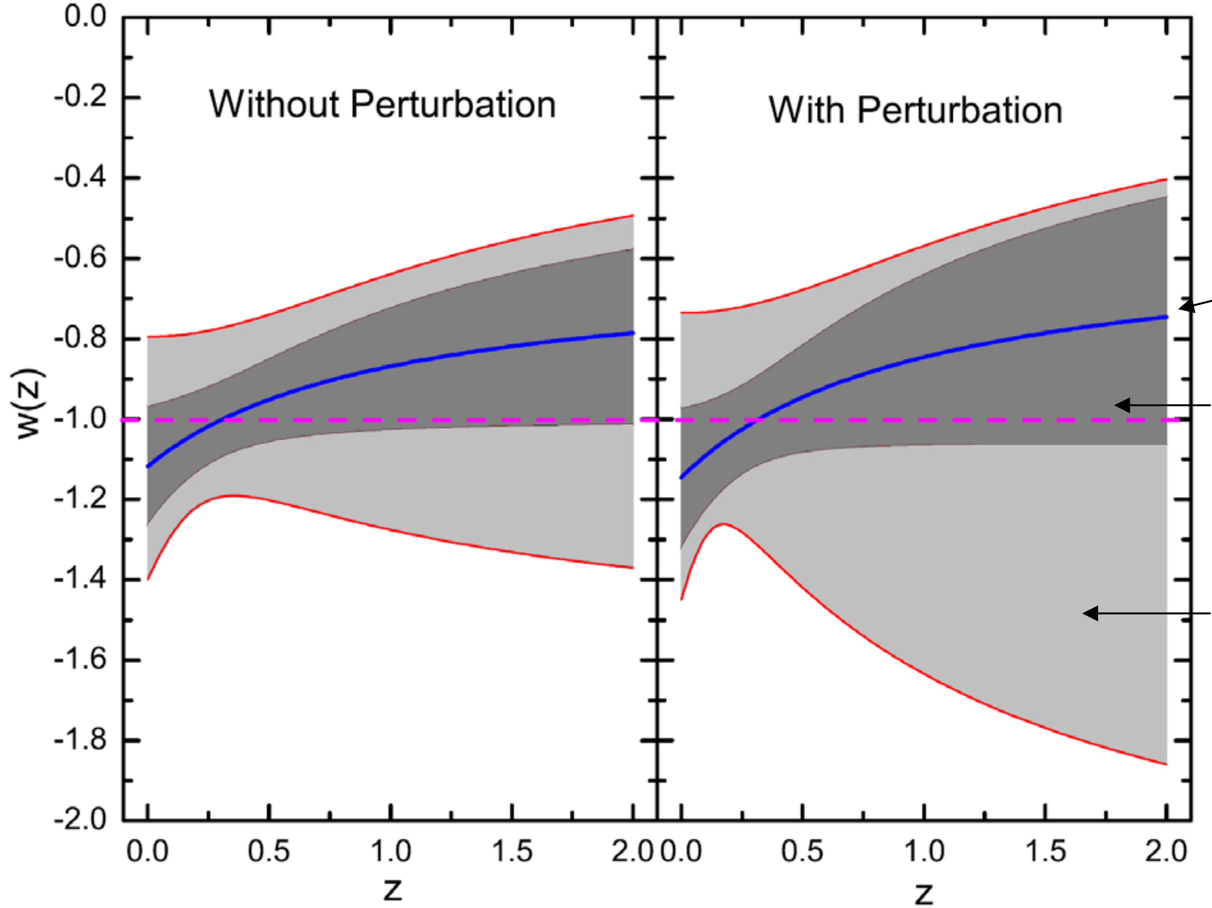
SNLS data set+CMB+BAO

- $\Omega_{0\mathrm{m}} = 0.28 \pm 0.03$
- 2σ confidence level.

< Data fitting of $w(z)$ (3) >

$$w(z) = w_0 + w_1 \frac{z}{1+z}$$

From [Zhao, Xia, Feng and Zhang, Int. J. Mod. Phys. D 16, 1229 (2007) [arXiv:astro-ph/0603621]]



Best-fit

68% **confidence level**

95% **confidence level**

157 "gold" SN Ia data set+WMAP 3-year data+SDSS with/without dark energy perturbations.